Problem Set 5

1. \[ A = (AB)(B^{-1}) \]
   - invertible (given)
   - invertible (since \(A\) has a inverse)

2. (a)
   \[
   \begin{bmatrix}
   2 & 1 & 1 \\
   6 & 4 & 5 \\
   4 & 1 & 3
   \end{bmatrix}
   \xrightarrow{-3R_1 + R_2} \begin{bmatrix}
   2 & 1 & 1 \\
   0 & 1 & 2 \\
   4 & 1 & 3
   \end{bmatrix}
   \xrightarrow{-2R_1 + R_3} \begin{bmatrix}
   2 & 1 & 1 \\
   0 & 1 & 2 \\
   0 & -1 & 1
   \end{bmatrix}
   \xrightarrow{R_2 + R_3} \begin{bmatrix}
   2 & 1 & 1 \\
   0 & 1 & 2 \\
   0 & 0 & 3
   \end{bmatrix}
   \text{ (upper triangular)}
   \]

   \[ E = \begin{bmatrix}
   1 & 0 & 0 \\
   -3 & 1 & 0 \\
   0 & 0 & 1
   \end{bmatrix} \quad F = \begin{bmatrix}
   1 & 0 & 0 \\
   0 & 1 & 0 \\
   -2 & 0 & 1
   \end{bmatrix} \quad G = \begin{bmatrix}
   1 & 0 & 0 \\
   0 & 1 & 0 \\
   0 & 1 & 1
   \end{bmatrix} \]

   (b) \[ E^T = \begin{bmatrix}
   1 & 0 & 0 \\
   3 & 1 & 0 \\
   0 & 0 & 1
   \end{bmatrix} \quad F^{-1} = \begin{bmatrix}
   1 & 0 & 0 \\
   0 & 1 & 0 \\
   2 & 0 & 1
   \end{bmatrix} \quad G^{-1} = \begin{bmatrix}
   1 & 0 & 0 \\
   0 & 1 & 0 \\
   0 & 0 & 1
   \end{bmatrix} \]

   \[ F^{-1}G^{-1} = \begin{bmatrix}
   1 & 0 & 0 \\
   0 & 1 & 0 \\
   2 & -1 & 1
   \end{bmatrix} \quad E^{-1}F^{-1}G^{-1} = \begin{bmatrix}
   1 & 0 & 0 \\
   3 & 1 & 0 \\
   2 & -1 & 1
   \end{bmatrix} \Rightarrow \text{ lower triang.} \]

(2R_1 + R_3 applied to \(G^{-1}\))

(3R_1 + R_2 applied to \(F^{-1}G^{-1}\))
Since $2\mathbf{a}_1 + 3\mathbf{a}_2 - 4\mathbf{a}_3 = \mathbf{0}$, it follows that $A\hat{x} = \hat{0}$ has more than one solution and hence infinitely many solutions. Thus the RREF of $[A; \hat{0}]$ has a free variable and hence a column without a pivot. Thus $A$ does not row-reduce to the identity matrix and so is not invertible.

Or: Since the columns of $A$ are linearly dependent, $A$ is not invertible, by the Invertible Matrix Theorem.
Option 1: AB is a $3 \times 4$ matrix. You can't have 4 linearly independent vectors in $\mathbb{R}^3$ and so the columns of $AB$ must be linearly dependent.

Option 2: Note that

$$AB = \begin{bmatrix} \bar{A}v_1 & \bar{A}v_2 & \bar{A}v_3 & \bar{A}v_4 \end{bmatrix}.$$  

Then

$$3\bar{A}v_1 - \bar{A}v_2 + 5\bar{A}v_3 + 7\bar{A}v_4 = A(3\bar{v}_1 - \bar{v}_2 + 5\bar{v}_3 + 7\bar{v}_4)$$  

since $A\bar{v}_x = A(x\bar{v})$

$$= 4\bar{0} = \bar{0}.$$  

Since a nontrivial linear combination of the columns of $AB$ yields the zero vector, these columns are linearly dependent.

6. (a)  

$$\begin{bmatrix} 5 & -1 & 12 \\ 1 & -2 & 1 \\ 1 & 4 & -1 \\ 2 & 1 & -5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$  

no free variables  

$\Rightarrow$ linearly independent

(b) 4 vectors in $\mathbb{R}^3$? Linearly dependent

(c) $\bar{v}_2$ is not a scalar multiple of $\bar{v}_1 \Rightarrow$ Linearly independent

(d) $\bar{v}_3 = 2\bar{v}_1 + \bar{v}_2 \Rightarrow$ Linearly dependent
\[
\begin{pmatrix}
1 & -2 & 2 \\
-3 & -2 & 4 \\
2 & 1 & -1
\end{pmatrix}
\xrightarrow{\text{RREF}}
\begin{pmatrix}
1 & 0 & 4/3 \\
0 & 1 & -1/3 \\
0 & 0 & 0
\end{pmatrix}
\]

This tells us that
\[
\vec{v}_3 = \frac{4}{3} \vec{v}_1 - \frac{1}{3} \vec{v}_2.
\]

7. Note that
\[
a_1 \vec{v}_1 + a_2 \vec{v}_2 + a_3 \vec{v}_3 = \vec{u} = b_1 \vec{v}_1 + b_2 \vec{v}_2 + b_3 \vec{v}_3
\]
\[\Rightarrow \vec{0} = (a_1 - b_1) \vec{v}_1 + (a_2 - b_2) \vec{v}_2 + (a_3 - b_3) \vec{v}_3\]
Since \(\vec{v}_1, \vec{v}_2, \vec{v}_3\) are linearly independent,
\[\Rightarrow a_1 - b_1 = a_2 - b_2 = a_3 - b_3\]
and so \(a_1 = b_1, a_2 = b_2, a_3 = b_3\).

8. Note that \(\vec{u}_2 = 2\vec{v}_1 + \vec{v}_2\).
\[\Rightarrow 2\vec{u}_1 + \vec{u}_2 - \vec{u}_3 + 0 \vec{u}_4 = 0\]
\[\Rightarrow \vec{u} \begin{bmatrix}
2 \\
1 \\
-1
\end{bmatrix} = \vec{0} \Rightarrow x = \begin{bmatrix}
\frac{2}{1} \\
\frac{1}{-1} \\
0
\end{bmatrix} \text{ is a solution}
\Rightarrow A\vec{x} = \vec{0}.
\]
Since \(U\) and \(A\) are row equivalent, \(U\vec{x} = \vec{0}\)
and \(A\vec{x} = \vec{0}\) have the exact same solutions.

Thus \(x = \begin{bmatrix}
\frac{2}{1} \\
\frac{1}{-1} \\
0
\end{bmatrix}\) is a solution \(\Rightarrow A\vec{x} = \vec{0} \Rightarrow A\begin{bmatrix}
\frac{3}{7} \\
0
\end{bmatrix} = \vec{0}\)
\[\Rightarrow 2\vec{a}_1 + \vec{a}_2 - 3\vec{a}_3 + 0\vec{a}_4 = 0 \Rightarrow \vec{a}_3 = 2\vec{a}_1 + \vec{a}_2.\]
tl;dr version: Since $A$ and $U$ are row equivalent, their columns have the same linear dependence relationships.

\[ a_3 = 2a_1 + a_2 \Rightarrow \begin{bmatrix} -\frac{3}{5} \\ \frac{2}{5} \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 4 \\ -3 \\ 7 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 \\ 7 \\ 11 \\ 3 \end{bmatrix} \]

\[ a_4 = a_1 + 4a_2 \Rightarrow \begin{bmatrix} -\frac{3}{5} \\ \frac{2}{5} \\ 2 \\ 1 \end{bmatrix} + 4 \begin{bmatrix} 4 \\ -3 \\ 7 \\ -1 \end{bmatrix} = \begin{bmatrix} 13 \\ -7 \\ 30 \\ -3 \end{bmatrix} \]