Example:
Find all integer solutions to the system
\[
\begin{align*}
  y &\equiv 2x + 3 \pmod{10} \\
  y &\equiv 4x + 1 \pmod{10}
\end{align*}
\]
Congruence is transitive, so
\[
2x + 3 \equiv 4x + 1 \pmod{10}
\]
\[
2 \equiv 2x \pmod{10}
\]
Thus \[x = \ldots, -8, 2, 12, 22, 32, \ldots\]
and \[x = \ldots, -4, 1, 6, 11, 16, \ldots\]
\[
x \text{ can be any of these numbers.}
\]
Suppose \[x = 1\]. Then
\[
\begin{align*}
  y &\equiv 2(1) + 3 \pmod{10} \\
  y &\equiv 5 \pmod{10} \\
  y &\equiv \ldots, -5, 5, 15, 25, \ldots
\end{align*}
\]
Suppose \[x = 6\]. Then
\[
\begin{align*}
  y &\equiv 2(6) + 3 \pmod{10} \\
  y &\equiv 15 \pmod{10} \\
  y &\equiv \ldots, -5, 5, 15, 25, \ldots
\end{align*}
\]
In general, if \[2x \equiv 2 \pmod{10}\],
then \[y \equiv 2 + 3 \pmod{10}\].
Thus, we have solutions:
\[(x, y) \text{ where } x = \ldots, -4, 1, 6, 11, 16, \ldots\]
and \[y = \ldots, -5, 5, 15, 25, \ldots\]
\[
\text{Note: If you “simplify”} \\
2 \equiv 2x \pmod{10}\]
to \[1 \equiv x \pmod{10}\]
you’ll get \[x = \ldots, 1, 11, 21, \ldots\]
and miss “half” the solutions. Dividing both sides of a congruence equation by a number that has a factor in common with the modulus doesn’t work.
Sample solutions:
\[(1, 5), (11, 15), (-4, 15), (6, 25), (116, 75)\]