Math 216 Spring 2012
Problem Set 3 Answer Key

1. The random variable X is the score on one roll of two six-sided dice, and subtracting the smaller number from the larger one. That means our possible scores are:

	1	2	3	4	5	6
1	0	1	2	3	4	5
2	1	0	1	2	3	4
3	2	1	0	1	2	3
4	3	2	1	0	1	2
5	4	3	2	1	0	1
6	5	4	3	2	1	0

   (a) Probability distribution function for X:

x	$f(x)$
0	$6/36 = 1/6$
1	$10/36 = 5/18$
2	$8/36 = 2/9$
3	$6/36 = 1/6$
4	$4/36 = 1/9$
5	$2/36 = 1/18$
otherwise	0

   (b) $E(X) = \frac{1}{6}(0) + \frac{5}{18}(1) + \frac{2}{9}(2) + \frac{1}{6}(3) + \frac{1}{9}(4) + \frac{1}{18}(5) = \frac{35}{18} \approx \mathbf{1.944}$

   (c) $\mathrm{Var}(X) = \frac{1}{6}(0-\frac{35}{18})^2 + \frac{5}{18}(1-\frac{35}{18})^2 + \frac{2}{9}(2-\frac{35}{18})^2 + \frac{1}{6}(3-\frac{35}{18})^2 + \frac{1}{9}(4-\frac{35}{18})^2 + \frac{1}{18}(5-\frac{35}{18})^2 = 665/324 \approx \mathbf{2.052}$

2. A company has five warehouses, only two of which have a particular product in stock. A salesperson calls the five warehouses in random order until a warehouse with the product is reached. Let the random variable X be the number of calls made by the salesperson.

   (a) The salesperson will make the first call. The probability that he will succeed (therefore not make any further calls) is 2/5. If he fails, he moves on to the second call, where his probability of success is now 2/4. Hence, the probability that he makes only two calls (stops on the second call) is 3/5 (the probability of failing on the first call) multiplied by 2/4. Similarly, the probability that he makes three calls is the probability of failing on the first call (3/5) times the probability of failing on the second call (2/4) times the probability of success on the third call (2/3). If he fails on the first three calls, his probability of success on the fourth call is 2/2. Therefore, he will make between one and four calls.

Probability distribution function for X:

x	$f(x)$
1	$\mathbf{2/5}$
2	$3/5 \cdot 1/2 = \mathbf{3/10}$
3	$3/5 \cdot 1/2 \cdot 2/3 = \mathbf{1/5}$
4	$3/5 \cdot 1/2 \cdot 1/3 \cdot 1 = \mathbf{1/10}$
otherwise	0

(b) $E(X) = 1(\frac{2}{5}) + 2(\frac{3}{10}) + 3\frac{1}{5} + 4\frac{1}{10} = \mathbf{2}$

3. (a) Confirm $f(x) \geq 0$ for all x: Since we are given that $f(x) = 0$ for all $x < 0$, we only have to show $f(x) \geq 0$ for $x \geq 0$. For nonnegative values of x, the denominator of the function is positive, therefore $f(x)$ is positive.
Confirm $\int_{-\infty}^{\infty} f(x)dx = 1$:

$$\begin{aligned}\int_{-\infty}^{\infty} f(x)dx &= \int_{-\infty}^{0} 0dx + \int_{0}^{\infty} \frac{2}{(x+1)^3}dx \\ &= 0 + \left.\frac{-1}{(x+1)^2}\right|_0^\infty \\ &= \lim_{x\to\infty} \frac{-1}{(x+1)^2} - (-1) = 0 + 1 = 1\square\end{aligned}$$

(b) Probability of failure in the first five hours:

$$\int_0^5 f(x)dx = \frac{-1}{6^2} - (-1) = \mathbf{\frac{35}{36}}$$

(c) Probability of lasting at least five hours:

$$\int_5^\infty f(x)dx = 1 - \int_0^5 f(x)dx = 1 - \frac{35}{36} = \mathbf{\frac{1}{36}}$$

4. (a) Probability density function for θ:

$$f(\theta) = \begin{cases} 1/180 & 0 \leq \theta \leq 180 \\ 0 & \text{otherwise} \end{cases}$$

(b) Probability that the minute hand will end up between 45 and 90 degrees:

$$\int_{45}^{90} f(\theta)d\theta = \left.\frac{\theta}{180}\right|_{45}^{90} = \frac{90}{180} - \frac{45}{180} = \mathbf{\frac{1}{4}}$$

5. X: radius of piston head, $\mu_X = 30$, $\sigma_X = 0.05$
Y: inside radius of cylinder, $\mu_Y = 30.25$, $\sigma_Y = 0.06$

(a) Define random variable $G = Y - X$. Then $E(G) = E(Y - X) = E(Y) - E(X) = 30 - 30.25 = \mathbf{0.25}$

(b) Stdev(G) = stdev($Y - X$) = $\sqrt{\sigma_Y^2 + \sigma_X^2} = \mathbf{0.078}$

(c) (Note: The question defines the "gap" to be the random variable Z, but we will stick with the letter G, to avoid confusion between the random variable and the standard normal "z-score.")
Probability that the gap G is negative:

$$P(G < 0) = P\left(\text{z} < \frac{0 - \mu_G}{\sigma_G}\right) = P\left(\text{z} < \frac{-.25}{.078}\right) = P\left(\text{z} < -3.21\right) = \mathbf{.00069}$$

6. $\mu = 3, \sigma = 0.12$

(a) Probability that glass sheet is thicker than 3.2:

$$P(x > 3.2) = P\left(\text{z} > \frac{x - \mu}{\sigma}\right) = P\left(\text{z} > \frac{3.2 - 3}{0.12}\right)$$
$$= P(\text{z} > 1.67) = 1 - P(\text{z} < 1.67) = 1 - 0.9525 = \mathbf{0.0475}$$

(b) Probability that glass sheet is thinner than 2.7:

$$P(x < 3.2) = P\left(\text{z} < \frac{x - \mu}{\sigma}\right) = P\left(\text{z} < \frac{2.7 - 3}{0.12}\right)$$
$$= P(\text{z} < 2.5) = \mathbf{0.00621}$$

(c) Want to find c such that $P(3 - c \leq x \leq 3 + c) = 0.99$. Since our chart gives us areas to the left of x-values, and the standard normal distribution is symmetric, we can look for the value c such that $P(x < 3 - c) = 0.05$. From the chart, the z-score that corresponds to area of 0.05 is -2.575. Convert the z-score back to x, the thickness of glass:

$$-2.575 = \frac{x - 3}{.12}$$

Solve for x: $x = 2.691$
This give us the lower bound on that interval $[3 - c, 3 + c]$. Solve for c:
$2.691 = 3 - c \implies \mathbf{c = 0.309}$

7. Let X be the weights of bags. We want to find $\bar{x}$ such that $P(x < 10) = 0.02$. Using the standard normal distribution chart, we find that 0.02 corresponds with a z-score of -2.055. In other words,

$$P\left(\frac{x - \bar{x}}{.05} < -2.055\right) = .02$$

Solve: $\bar{x} = \mathbf{10.103}$.

8. The distribution is approximately normal, because the normal probability plot is close to a straight line. However, to receive full credit, you must also point out that the distribution deviates from normal because it is right-skewed, the normal probability plot is slightly concave up, or there is an outlier to the far right. You can also point out that we have limited information because the sample size is so small.