Math 194
Unit 5 Practice

1. Suppose $\mathbf{x}$ is an eigenvector for the $n \times n$ matrices A and B. Must $\mathbf{x}$ be an eigenvector for the matrix $A+B$? Justify your answer.

2. Suppose that λ is an eigenvalue for the $n \times n$ matrix A with eigenvector $\mathbf{x}$. Prove that the matrix $(A - \lambda I)$ is not invertible.

3. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that reflects vectors across the line $y = 3x$ and then scales them by a factor of 2. Find the eigenvalues for the standard matrix for T. For each eigenvalue, find a basis for its eigenspace.

4. Consider the matrix $A = \begin{bmatrix} 1 & k \\ 1 & 1 \end{bmatrix}$.

   (a) For which values of k does A have two distinct real eigenvalues?

   (b) For which values of k does A have only one eigenvalue?

5. Suppose that the 3×3 matrix A has the property that $\mathbf{a}_3 = 2\mathbf{a}_1 - 3\mathbf{a}_2$. Find the determinant of A.

6. Find the determinant of $A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}$ without using a calculator of any kind.

7. Consider the interaction between two populations, a population of foxes (the predators) and a population of rabbits (the prey). Denote the fox and rabbit populations at time k by $\mathbf{x}_k = \begin{bmatrix} F_k \\ R_k \end{bmatrix}$, where k is in years, F_k is the number of foxes after k years, and R_k is the number of rabbits after k years. Suppose that

$$\begin{aligned} F_{k+1} &= 0.86F_k + 0.08R_k \\ R_{k+1} &= -0.12F_k + 1.14R_k. \end{aligned}$$

   (a) Find closed formulas (that is, ones that don't involve recursion as the formulas above do) for the fox and rabbit populations at time k assuming $F(0) = 100$ and $R(0) = 800$.

   (b) Determine the long term growth (or decline) rate in the fox and rabbit populations.

   (c) Determine the long term proportion of foxes to rabbits.

8. There are three towns that I am totally making up in a very isolated part of England: Hartnell, Troughton, and Pertwee.

   Each year, 10% of Hartnell residents move to Troughton, 8% move to Pertwee, and the rest remain in Hartwell. Moreover, there is one child born in Hartnell for every 20 residents the year before. (These children don't move from Hartnell the year they're born.)

   Similarly, each year 5% of Troughton residents move to Hartnell, 11% move to Pertwee, and the rest remain in Troughton. The birth rate in Troughton is 8%. (That is, there are 8 babies born in Troughton each year for every 100 residents. These babies stay in Troughton the year they're born.)

   Over at Pertwee, 15% of residents die each year (it is the local retirement community), 12% move to Hartnell, 5% move to Troughton, and the rest remain in Pertwee. No children are born in Pertwee.

(a) Determine the long term growth (or decline) rate in the populations of these three towns.

(b) Determine the long term proportion of residents in these three towns.

1. Yes

2. Hint: Show that the vector $\mathbf{x}$ is in the null space of $(A - \lambda I)$.

3. $\lambda_1 = 2$ with basis $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$, $\lambda_2 = -2$ with basis $\begin{bmatrix} -3 \\ 1 \end{bmatrix}$

4. (a) $k > 0$
   (b) $k = 0$

5. $\det A = 0$

6. $\det A = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120$

7. (a) $F(k) = 300(1.1)^k - 200(0.9)^k$, $R(k) = 900(1.1)^k - 100(0.9)^k$
   (b) 10% growth
   (c) $1 : 3$

8. (a) 1.9% growth
   (b) Approximately $55 : 75 : 37$