Math 194
Unit 4 Practice

1. Determine whether each of the following transformations is linear. Justify your answers.

   (a) $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T(\mathbf{x}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

   (b) $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T(\mathbf{x}) = \begin{bmatrix} x_2^2 \\ x_3 \end{bmatrix}$

   (c) $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(\mathbf{x}) = \begin{bmatrix} x_1 + x_2 \\ x_1 - x_2 \end{bmatrix}$

   (d) $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(\mathbf{x}) = \begin{bmatrix} x_2 - x_3 \\ x_1 x_3 \\ x_1 - x_2 \end{bmatrix}$

2. Describe geometrically the effects of each of the following linear transformations.

   (a) $T : \mathbb{R}^2 \to \mathbb{R}^2$ with standard matrix $\begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$

   (b) $T : \mathbb{R}^2 \to \mathbb{R}^2$ with standard matrix $\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$

3. Suppose that $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation, $\mathbf{u} = \begin{bmatrix} 5 \\ 42 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 6 \\ 41 \end{bmatrix}$, $T(\mathbf{u}) = \begin{bmatrix} 89 \\ 52 \end{bmatrix}$, and $T(\mathbf{v}) = \begin{bmatrix} 88 \\ 53 \end{bmatrix}$. Find the standard matrix for T.

4. The conversion formula $C = \frac{5}{9}(F - 32)$ from Fahrenheit to Celsius is nonlinear, but we can use the idea of homogeneous coordinates to represent this conversion with a linear transformation.

   (a) Find the 2×2 matrix A that transforms the vector $\begin{bmatrix} F \\ 1 \end{bmatrix}$ into the vector $\begin{bmatrix} C \\ 1 \end{bmatrix}$.

   (b) Find the inverse of the matrix you found in part (a) and confirm that it converts Celsius temperatures into Fahrenheit temperatures.

5. Find the standard matrix for each of the following linear transformations.

   (a) $T : \mathbb{R}^2 \to \mathbb{R}^2$ dilates a vector by a factor of 3, then reflects that vector about the line $y = x$, and then projects that vector onto the y-axis

   (b) $T : \mathbb{R}^3 \to \mathbb{R}^3$ projects a vector onto the xz-plane and then projects that vector onto the xy-plane

6. Find the standard matrix for a shear in the x-direction that transforms the triangle with vertices $(0, 0)$, $(2, 1)$, and $(3, 0)$ into a right triangle with a right angle at the origin.

7. Which of the following linear transformations are one-to-one? Which are onto?

   (a) $T : \mathbb{R}^2 \to \mathbb{R}^2$ is the dilation $T(\mathbf{x}) = 2\mathbf{x}$

(b) $T : \mathbb{R}^2 \to \mathbb{R}^2$ has standard matrix $\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$

(c) $T : \mathbb{R}^3 \to \mathbb{R}^3$ projects a vector onto the yz-plane

(d) $T : \mathbb{R}^3 \to \mathbb{R}^3$ reflects vectors through the xy-plane

(e) $T : \mathbb{R}^3 \to \mathbb{R}^2$ has standard matrix $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \end{bmatrix}$

8. Consider the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ with standard matrix $\begin{bmatrix} 1 & 5 & 4 \\ -2 & -1 & 1 \\ 1 & 3 & 2 \end{bmatrix}$. Show that this transformation is not onto, then find a vector that is not in its range.

9. For each of the following, determine if $T_1 \circ T_2 = T_2 \circ T_1$.

   (a) $T_1 : \mathbb{R}^2 \to \mathbb{R}^2$ is the projection onto the x-axis and $T_2 : \mathbb{R}^2 \to \mathbb{R}^2$ is the projection onto the y-axis

   (b) $T_1 : \mathbb{R}^2 \to \mathbb{R}^2$ is the rotation clockwise about the origin by 45 degrees and $T_2 : \mathbb{R}^2 \to \mathbb{R}^2$ is the rotation clockwise about the origin by 90 degrees

   (c) $T_1 : \mathbb{R}^2 \to \mathbb{R}^2$ is the rotation clockwise about the origin by 90 degrees and $T_2 : \mathbb{R}^2 \to \mathbb{R}^2$ is the reflection about the line $y = 3x$

10. Suppose that $T_1 : \mathbb{R}^n \to \mathbb{R}^m$ and $T_2 : \mathbb{R}^m \to \mathbb{R}^k$ are linear transformations and that T_1 is not one-to-one. Prove that the composition $T_2 \circ T_1$ is also not one-to-one.

11. Find the 3×3 matrix that produces the following two-dimensional transformation, using homogeneous coordinates: T reflects points across the line $y = x - 2$.

12. Suppose that $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, that $\mathbf{u}$ and $\mathbf{v}$ are linearly independent vectors in $\mathbb{R}^n$, and that $T(\mathbf{u})$ and $T(\mathbf{v})$ are linearly dependent vectors in $\mathbb{R}^m$. Prove that T is not one-to-one.

1. (a) Linear
   (b) Not linear
   (c) Linear
   (d) Not linear
2. (a) Rotation clockwise about the origin by 135°
   (b) Reflection about the line through the origin with positive slope that makes a 22.5° angle with the x-axis
3. $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$
4. (a) $\begin{bmatrix} \frac{5}{9} & -\frac{160}{9} \\ 0 & 1 \end{bmatrix}$
   (b) $\begin{bmatrix} \frac{9}{5} & 32 \\ 0 & 1 \end{bmatrix}$
5. (a) $\begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix}$
   (b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$
6. $\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$
7. (a) One-to-one, onto
   (b) Not one-to-one, not onto
   (c) Not one-to-one, not onto
   (d) One-to-one, onto
   (e) Not one-to-one, onto
8. Hint: Recall that the range of a linear transformation is equal to its column space. Note that the third column of the standard matrix is a linear combination of the first two. Find a vector that is not a linear combination of the first two columns.
9. (a) Yes
   (b) Yes
   (c) No
10. Hint: Since T_1 is not one-to-one, you can find two vectors in $\mathbb{R}^n$ that map to the same vector in $\mathbb{R}^m$. Use these two vectors to show that $T_2 \circ T_1$ is not one-to-one.
11. $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & -2 \\ 0 & 0 & 1 \end{bmatrix}$
12. Hint: Since $T(\mathbf{u})$ and $T(\mathbf{v})$ are linearly dependent, there is some linear combination of these two vectors that equals the zero vector.