PROBLEM SET 5

① $A = (AB)(B^{-1})$

$\underbrace{(AB)}_{\text{invertible (given)}}$ $\underbrace{(B^{-1})}_{\text{invertible (since it's an inverse)}}$

Since the product of two invertible matrices is invertible, A is invertible.

② (a)

$$\begin{bmatrix} 2 & 1 & 1 \\ 6 & 4 & 5 \\ 4 & 1 & 3 \end{bmatrix} \xrightarrow[(E)]{-3R_1+R_2} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix}$$

$$\xrightarrow[(F)]{-2R_1+R_3} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & -1 & 1 \end{bmatrix} \xrightarrow[(G)]{R_2+R_3} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

upper triangular (U)

$$E = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \quad G = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

(b)

$$E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad F^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \quad G^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$$F^{-1}G^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix} \qquad E^{-1}F^{-1}G^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix} = L$$

($2R_1 + R_3$ applied to G^{-1})

($3R_1 + R_2$ applied to $F^{-1}G^{-1}$)

lower triang.

(3)

$$\left[\begin{array}{ccc|ccc} -1 & -3 & -3 & 1 & 0 & 0 \\ 2 & 6 & 1 & 0 & 1 & 0 \\ 3 & 8 & 3 & 0 & 0 & 1 \end{array}\right] \xrightarrow[-R_1]{} \left[\begin{array}{ccc|ccc} 1 & 3 & 3 & -1 & 0 & 0 \\ 2 & 6 & 1 & 0 & 1 & 0 \\ 3 & 8 & 3 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[\substack{-2R_1+R_2 \\ -3R_1+R_3}]{} \left[\begin{array}{ccc|ccc} 1 & 3 & 3 & -1 & 0 & 0 \\ 0 & 0 & -5 & 2 & 1 & 0 \\ 0 & -1 & -6 & 3 & 0 & 1 \end{array}\right] \xrightarrow[R_2 \leftrightarrow R_3]{} \left[\begin{array}{ccc|ccc} 1 & 3 & 3 & -1 & 0 & 0 \\ 0 & -1 & -6 & 3 & 0 & 1 \\ 0 & 0 & -5 & 2 & 1 & 0 \end{array}\right]$$

$$\xrightarrow[\substack{-R_2 \\ -\frac{1}{5}R_3}]{} \left[\begin{array}{ccc|ccc} 1 & 3 & 3 & -1 & 0 & 0 \\ 0 & 1 & 6 & -3 & 0 & -1 \\ 0 & 0 & 1 & -2/5 & -1/5 & 0 \end{array}\right] \xrightarrow[\substack{-6R_3+R_2 \\ -3R_3+R_1}]{} \left[\begin{array}{ccc|ccc} 1 & 3 & 0 & 1/5 & 3/5 & 0 \\ 0 & 1 & 0 & -3/5 & 6/5 & -1 \\ 0 & 0 & 1 & -2/5 & -1/5 & 0 \end{array}\right]$$

$$\xrightarrow[-3R_2+R_1]{} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -3 & 3 \\ 0 & 1 & 0 & -3/5 & 6/5 & -1 \\ 0 & 0 & 1 & -2/5 & -1/5 & 0 \end{array}\right] \qquad A^{-1} = \begin{bmatrix} 2 & -3 & 3 \\ -3/5 & 6/5 & -1 \\ -2/5 & -1/5 & 0 \end{bmatrix}$$

(4) Since $2\vec{a}_1 + \vec{a}_2 - 4\vec{a}_3 = \vec{0}$, it follows that $A\vec{x} = \vec{0}$ has more than one solution and thus infinitely many solutions. Thus the RREF of $[A \mid \vec{0}]$ has a free variable and hence a column _without_ a pivot. Thus A does _not_ row-reduce to the identity matrix and so _isn't_ invertible.

_Or_: Since the columns of A are linearly dependent, A is not invertible, by the Invertible Matrix Theorem.

(5) Option 1: AB is a 3×4 matrix. You can't have 4 linearly independent vectors in $\mathbb{R}^3$ and so the columns of AB must be linearly dependent.

Option 2: Note that

$$AB = [A\vec{b}_1 \ A\vec{b}_2 \ A\vec{b}_3 \ A\vec{b}_4].$$

Then $3A\vec{b}_1 - A\vec{b}_2 + 5A\vec{b}_3 + 7Ab_4$

$= A(3\vec{b}_1 - \vec{b}_2 + 5\vec{b}_3 + 7\vec{b}_4)$ since $\alpha A\vec{x} = A(\alpha\vec{x})$

$= A\vec{0} = \vec{0}$.

Since a nontrivial linear combination of the columns of AB yields the zero vector, those columns are linearly dependent.

(6) (a) $\begin{bmatrix} 5 & -1 & 12 \\ 1 & -2 & 1 \\ 1 & 4 & -1 \\ 2 & 1 & -5 \end{bmatrix} \xrightarrow[\text{RREF}]{} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ no free variables $\Rightarrow$ linearly independent

(b) 4 vectors in $\mathbb{R}^3$? linearly dependent

(c) $\vec{v}_2$ is not a scalar multiple of $\vec{v}_1 \Rightarrow$ linearly independent

(d) $\vec{v}_3 = 2\vec{v}_1 + \vec{v}_2 \Rightarrow$ linearly dependent

(e) $\begin{bmatrix} 1 & -2 & 2 \\ -3 & -24 & 4 \\ 2 & 11 & -1 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 4/3 \\ 0 & 1 & -1/3 \\ 0 & 0 & 0 \end{bmatrix}$ free variable $\Rightarrow$ linearly dependent

This tells us that
$\vec{v}_3 = \frac{4}{3}\vec{v}_1 - \frac{1}{3}\vec{v}_2$.

7. Note that

$$a_1\vec{v}_1 + a_2\vec{v}_2 + a_3\vec{v}_3 = \vec{u} = b_1\vec{v}_1 + b_2\vec{v}_2 + b_3\vec{v}_3$$

$$\Rightarrow \vec{0} = (a_1-b_1)\vec{v}_1 + (a_2-b_2)\vec{v}_2 + (a_3-b_3)\vec{v}_3$$

Since $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are linearly independent, it follows that $0 = a_1-b_1 = a_2-b_2 = a_3-b_3$

and so $a_1 = b_1$, $a_2 = b_2$, $a_3 = b_3$.

8. Note that $\vec{u}_3 = 2\vec{u}_1 + \vec{u}_2$. $\Rightarrow 2\vec{u}_1 + \vec{u}_2 - \vec{u}_3 + 0\vec{u}_4 = 0$

$$\Rightarrow U\begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix} = \vec{0} \Rightarrow \vec{x} = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix}$$ is a solution to $U\vec{x} = \vec{0}$.

Since U and A are row equivalent, $U\vec{x} = \vec{0}$ and $A\vec{x} = \vec{0}$ have the exact same solutions.

Thus $\vec{x} = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix}$ is a solution to $A\vec{x} = \vec{0} \Rightarrow A\begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix} = \vec{0}$

$$\Rightarrow 2\vec{a}_1 + \vec{a}_2 - \vec{a}_3 + 0\vec{a}_4 = 0 \Rightarrow \vec{a}_3 = 2\vec{a}_1 + \vec{a}_2.$$

tl;dr version: Since A and U are row equivalent, their columns have the same linear dependence relationships.

$$\Rightarrow \vec{a}_3 = 2\vec{a}_1 + \vec{a}_2 = 2\begin{bmatrix}-3\\5\\2\\1\end{bmatrix} + \begin{bmatrix}4\\-3\\7\\-1\end{bmatrix} = \begin{bmatrix}-2\\7\\11\\1\end{bmatrix}$$

$$\vec{a}_4 = \vec{a}_1 + 4\vec{a}_2 = \begin{bmatrix}-3\\5\\2\\1\end{bmatrix} + 4\begin{bmatrix}4\\-3\\7\\-1\end{bmatrix} = \begin{bmatrix}13\\-7\\30\\-3\end{bmatrix}$$