PROBLEM SET 7

① (a) Both the +2 in the first component and the absolute value in the second component pose a problem. Consider $\alpha = -2$ and $\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$$\alpha T(\vec{x}) = -2\, T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = -2 \begin{bmatrix} 4(1)-2 \\ 3|1| \end{bmatrix} = -2 \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -4 \\ -6 \end{bmatrix}$$

but $T(\alpha\vec{x}) = T\left(-2\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = T\left(\begin{bmatrix} -2 \\ -2 \end{bmatrix}\right) = \begin{bmatrix} 4(-2)-2 \\ 3|-2| \end{bmatrix}$

$$= \begin{bmatrix} -10 \\ 6 \end{bmatrix} \neq \alpha T(\vec{x}).$$ NOT LINEAR.

(b) Let $\alpha \in \mathbb{R}$ and $\vec{x} \in \mathbb{R}^2$. Then

$$\alpha T(\vec{x}) = \alpha \begin{bmatrix} 3x_1 + 5x_2 \\ 4x_1 - x_2 \end{bmatrix} = \begin{bmatrix} \alpha(3x_1 + 5x_2) \\ \alpha(4x_1 - x_2) \end{bmatrix}$$

$$= \begin{bmatrix} 3\alpha x_1 + 5\alpha x_2 \\ 4\alpha x_1 - \alpha x_2 \end{bmatrix}.$$

Also, $T(\alpha\vec{x}) = T\left(\begin{bmatrix} \alpha x_1 \\ \alpha x_2 \end{bmatrix}\right) = \begin{bmatrix} 3\alpha x_1 + 5\alpha x_2 \\ 4\alpha x_1 - \alpha x_2 \end{bmatrix}$

$$\Rightarrow \alpha T(\vec{x}) = T(\alpha\vec{x}). \checkmark$$

Let $y \in \mathbb{R}^2$. Then

$$T(\vec{x}+\vec{y}) = T\left(\begin{bmatrix} x_1+y_1 \\ x_2+y_2 \end{bmatrix}\right) = \begin{bmatrix} 3(x_1+y_1) + 5(x_2+y_2) \\ 4(x_1+y_1) - (x_2+y_2) \end{bmatrix}$$

$$= \begin{bmatrix} 3x_1 + 3y_1 + 5x_2 + 5y_2 \\ 4x_1 + 4y_1 - x_2 - y_2 \end{bmatrix}.$$

Also, $T(\vec{x})+T(\vec{y}) = \begin{bmatrix}3x_1+5x_2\\4x_1-x_2\end{bmatrix}+\begin{bmatrix}3y_1+5y_2\\4y_1-y_2\end{bmatrix}$

$$= \begin{bmatrix}3x_1+3y_1+5x_2+5y_2\\4x_1+4y_1-x_2-y_2\end{bmatrix} = T(\vec{x}+\vec{y}). \checkmark$$

LINEAR.

(c) The $+4$ in the second component and the -2 in the third are problems. Consider $\vec{x}=\begin{bmatrix}1\\2\end{bmatrix}$ and $\vec{y}=\begin{bmatrix}3\\4\end{bmatrix}$. Then

$$T(\vec{x}+\vec{y}) = T\left(\begin{bmatrix}4\\6\end{bmatrix}\right) = \begin{bmatrix}2(4)-3(6)\\4+4\\5(6)-2\end{bmatrix} = \begin{bmatrix}-10\\8\\28\end{bmatrix}$$

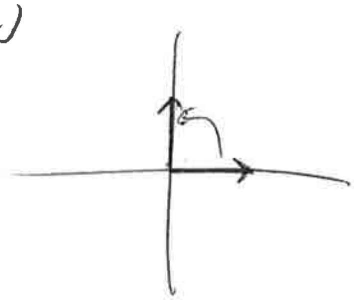
but $T(\vec{x})+T(\vec{y}) = \begin{bmatrix}2(1)-3(2)\\1+4\\5(2)-2\end{bmatrix}+\begin{bmatrix}2(3)-3(4)\\3+4\\5(4)-2\end{bmatrix}$

$$= \begin{bmatrix}-4\\5\\8\end{bmatrix}+\begin{bmatrix}-6\\12\\18\end{bmatrix} = \begin{bmatrix}-10\\17\\28\end{bmatrix}$$

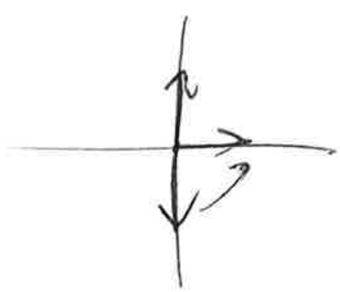
NOT LINEAR

Lucky coincidence!

(2) (a)



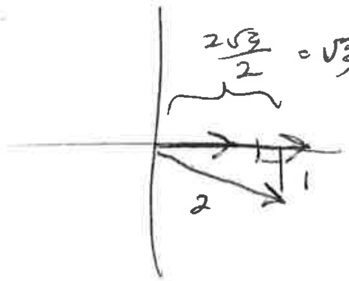
$\begin{bmatrix}1\\0\end{bmatrix} \xrightarrow{\text{refl.}} \begin{bmatrix}1\\0\end{bmatrix} \xrightarrow{\text{rot.}} \begin{bmatrix}0\\1\end{bmatrix}$

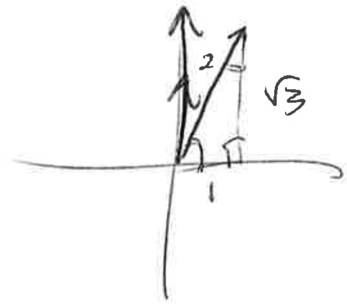


$\begin{bmatrix}0\\1\end{bmatrix} \xrightarrow{\text{refl.}} \begin{bmatrix}0\\-1\end{bmatrix} \xrightarrow{\text{rot.}} \begin{bmatrix}1\\0\end{bmatrix}$

$$\begin{bmatrix}0&1\\1&0\end{bmatrix}$$

(b)

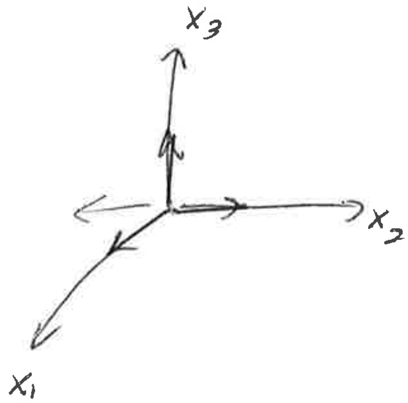




$$\begin{bmatrix} \sqrt{3} & 1 \\ -1 & \sqrt{3} \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \xrightarrow[\text{double}]{} \begin{bmatrix} 2 \\ 0 \end{bmatrix} \xrightarrow[\text{rot.}]{} \begin{bmatrix} \sqrt{3} \\ -1 \end{bmatrix} \qquad \begin{bmatrix} 0 \\ 1 \end{bmatrix} \xrightarrow[\text{double}]{} \begin{bmatrix} 0 \\ 2 \end{bmatrix} \xrightarrow[\text{rot.}]{} \begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix}$$

③ (a)



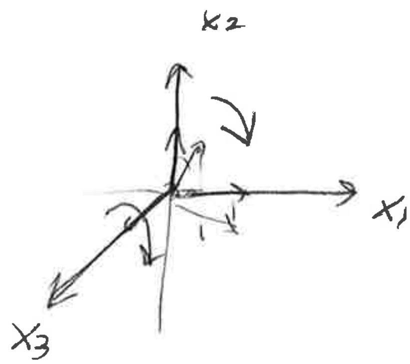
$$T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(b)



$$T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} \sqrt{3}/2 \\ -1/2 \\ 0 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1/2 \\ \sqrt{3}/2 \\ 0 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} \sqrt{3}/2 & 1/2 & 0 \\ -1/2 & \sqrt{3}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

④ $\begin{bmatrix} T(\vec{v}_1) & T(\vec{v}_2) & T(\vec{v}_3) \end{bmatrix} = \begin{bmatrix} 3 & 1 & 2 \\ 0 & 1 & 2 \\ -1 & 1 & 0 \end{bmatrix}$

$\parallel$

$\begin{bmatrix} A\vec{v}_1 & A\vec{v}_2 & A\vec{v}_3 \end{bmatrix}$ where A is the standard matrix for T.

$\parallel$

AV where $V = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ -1 & 1 & 2 \end{bmatrix}$

$$\Rightarrow A = \begin{bmatrix} 3 & 1 & 2 \\ 0 & 1 & 2 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ -1 & 1 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 3 & 1 & 2 \\ 0 & 1 & 2 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1/6 & 1/2 & -5/6 \\ 1/2 & -1/2 & 1/2 \\ -1/6 & 1/2 & -1/6 \end{bmatrix}$$

$$= \begin{bmatrix} 2/3 & 2 & -7/3 \\ 1/6 & 1/2 & 1/6 \\ 1/3 & -1 & 4/3 \end{bmatrix}$$

OR Note that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ form a basis for $\mathbb{R}^3$.

Then, for instance, $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3$

for some c_1, c_2, c_3.

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 1 & 3 & 0 \\ -1 & 1 & 2 & 0 \end{array}\right] \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 & 1/6 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & -1/6 \end{bmatrix}.$$

Thus $T(\vec{e}_1) = T\left(\tfrac{1}{6}\vec{v}_1 + \tfrac{1}{2}\vec{v}_2 - \tfrac{1}{6}\vec{v}_3\right)$

$= \frac{1}{6}T(\vec{v}_1) + \frac{1}{2}T(\vec{v}_2) - \frac{1}{6}T(\vec{v}_3)$

$= \frac{1}{6}\begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{6}\begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 1/6 \\ 1/3 \end{bmatrix}$ ← 1st column of A

you can use the same process to find the 2nd and 3rd columns of A using $\vec{e}_2$ and $\vec{e}_3$.

⑤ (a) $\begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 1 \\ 1 & 3 & 2 \end{bmatrix} \xrightarrow[\text{RREF}]{} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ no row of zeros $\Rightarrow$ ONTO

no free vars $\Rightarrow$ 1-1

(b) $\begin{bmatrix} 2 & 3 & -1 \\ 1 & 1 & 2 \\ 1 & 2 & -3 \end{bmatrix} \xrightarrow[\text{RREF}]{} \begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & -5 \\ 0 & 0 & 0 \end{bmatrix}$ row of zeros $\Rightarrow$ NOT ONTO

free variable $\Rightarrow$ NOT 1-1

(c) $\mathbb{R}^3 \to \mathbb{R}^2$ tells us it's NOT 1-1.

(You can't have each input go to a unique output when there are "more" inputs than outputs.)

$\begin{bmatrix} 1 & -5 & 4 \\ 0 & 1 & -6 \end{bmatrix} \xrightarrow[\text{RREF}]{} \begin{bmatrix} 1 & 0 & -26 \\ 0 & 1 & -6 \end{bmatrix}$ no row of zeros $\Rightarrow$ ONTO

⑥ (a) $m \leq n$. For example if $T: \mathbb{R}^3 \to \mathbb{R}^4$, you can't "fill" a 4-dimensional space with a 3-dimensional one.

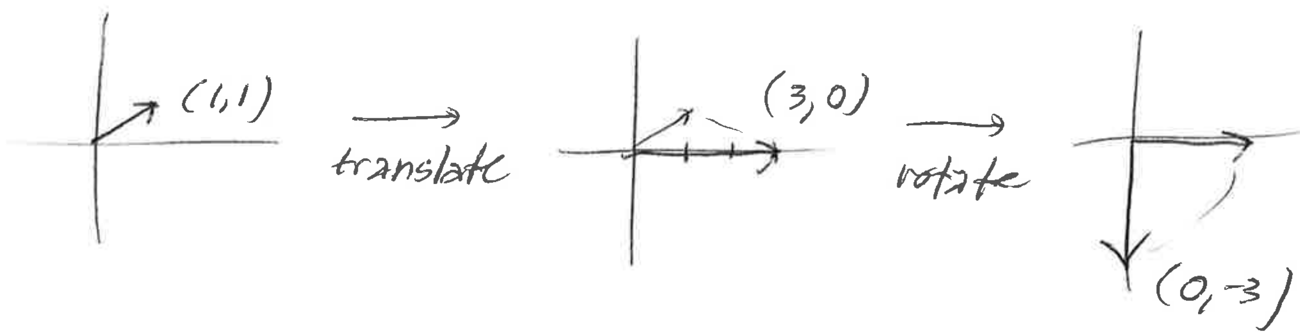
(b) $m \geq n$. For example, if $T: \mathbb{R}^4 \to \mathbb{R}^3$, you're guaranteed to get "overlaps."

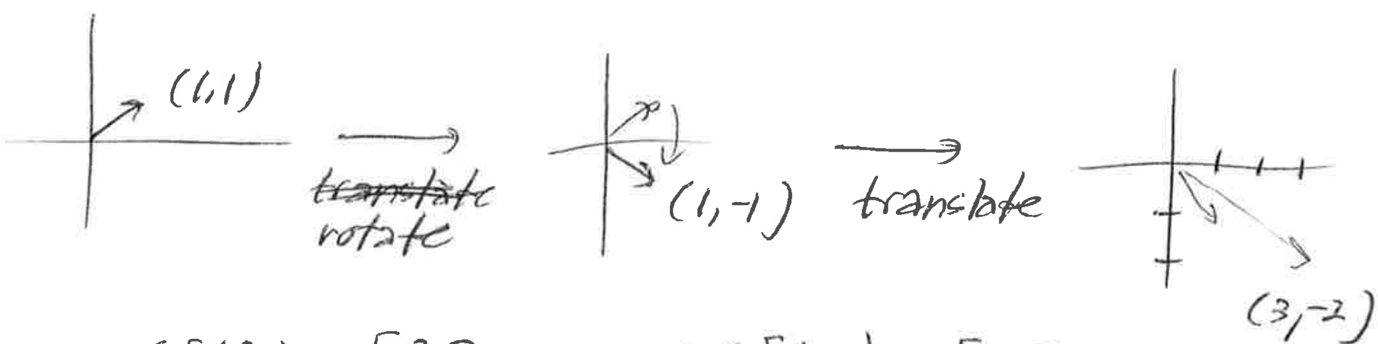
⑦ (a) Translation by 2 in the x-direction and −1 in the y-direction, <u>then</u> rotation by 90° CW about the origin.

(b) Vice versa.

(c) $AB = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & -2 \\ 0 & 0 & 1 \end{bmatrix}$ $BA = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}$ Nope.

You can also follow a particular vector. For instance, $\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$





$T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ -3 \end{bmatrix}$ but $L\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$

(8) Note that $T: \mathbb{R}^n \to \mathbb{R}^n$ and so has an $n \times n$ standard matrix A.

Since $T(\vec{u}) = T(\vec{v})$ for some pair of distinct vectors $\vec{u}$ and $\vec{v}$, T is not 1-1.

Thus the RREF of A has a free variable. This means it doesn't have a pivot in every column. Since it's square, this means it doesn't have a pivot in every row. This means it has a row of all zeros. Thus T is <u>not</u> onto.

Alternately,

T not 1-1 $\Rightarrow$ A is not invertible

$\Rightarrow$ T is not onto by the Invertible Matrix Theorem.