Math 194
Problem Set 7

1. For each of the following transformations, determine if the transformation is linear or not. Justify your answers.

   (a) $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(\mathbf{x}) = \begin{bmatrix} 4x_1 - 2 \\ 3|x_2| \end{bmatrix}$.

   (b) $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(\mathbf{x}) = \begin{bmatrix} 3x_1 + 5x_2 \\ 4x_1 - x_2 \end{bmatrix}$.

   (c) $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(\mathbf{x}) = \begin{bmatrix} 2x_1 - 3x_2 \\ x_1 + 4 \\ 5x_2 - 2 \end{bmatrix}$.

2. Find the standard matrix representation for each of the following linear transformations.

   (a) T is the transformation that reflects each vector in $\mathbb{R}^2$ about the x_1-axis and then rotates it 90° in the counterclockwise direction.

   (b) T doubles the length of each vector in $\mathbb{R}^2$ and then rotates it 30° in the clockwise direction.

3. (a) Find the standard matrix for the linear transformation that reflects vectors in $\mathbb{R}^3$ through the x_1-x_3 plane

   (b) Suppose that you are somewhere on the positive x_3-axis in $\mathbb{R}^3$ looking back toward the origin. The positive x_1-axis extends to your right, and the positive x_2-axis extends up. Find the standard matrix for the linear transformation that rotates vectors in $\mathbb{R}^3$ by 30 degrees clockwise about the x_3-axis given your orientation just described.

4. Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$. Note that these three vectors are linearly independent. Suppose that $T : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear transformation that has the following effects on these three vectors:

   $T(\mathbf{v}_1) = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}$, $T(\mathbf{v}_2) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, and $T(\mathbf{v}_3) = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$. Find the standard matrix representation for T.

5. Determine whether each of the following linear transformations is onto. Also determine whether each is one-to-one. Justify your answers.

   (a) $T : \mathbb{R}^3 \to \mathbb{R}^3$ with standard matrix $\begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 1 \\ 1 & 3 & 2 \end{bmatrix}$

   (b) $T : \mathbb{R}^3 \to \mathbb{R}^3$ with standard matrix $\begin{bmatrix} 2 & 3 & -1 \\ 1 & 1 & 2 \\ 1 & 2 & -3 \end{bmatrix}$

   (c) $T : \mathbb{R}^3 \to \mathbb{R}^2$ with standard matrix $\begin{bmatrix} 1 & -5 & 4 \\ 0 & 1 & -6 \end{bmatrix}$

6. Consider the linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$.

   (a) If T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$, what can you say about the relative sizes of m and n?

   (b) If T is one-to-one, what can you say about the relative sizes of m and n?

7. Suppose $A = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$.

   (a) Let T be the transformation with standard matrix AB. What 2D transformation is performed by T when it acts on the homogeneous coordinates of a point (x, y)?

   (b) Let L be the transformation with standard matrix BA. What 2D transformation is performed by T when it acts on the homogeneous coordinates of a point (x, y)?

   (c) Is the transformation performed by T the same as the transformation performed by L? Justify your answer.

8. Suppose a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ has the property that $T(\mathbf{u}) = T(\mathbf{v})$ for some pair of distinct vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$. Can T map $\mathbb{R}^n$ onto $\mathbb{R}^n$? Why or why not?