PROBLEM SET 6

$\vec{d} = \begin{bmatrix} 1.2 \\ 3.4 \\ 2.7 \\ 4.3 \\ 2.9 \end{bmatrix}$

① (a) $\vec{x} = C\vec{x} + \vec{d}$ ← production consumed externally

$\vec{x}$: total production

$C\vec{x}$: production consumed internally

$$I\vec{x} = C\vec{x} + \vec{d}$$
$$I\vec{x} - C\vec{x} = \vec{d}$$
$$(I - C)\vec{x} = \vec{d}$$
$$\vec{x} = (I-C)^{-1}\vec{d}$$

$$C = \begin{bmatrix} .27 & .39 & .03 & .02 & .23 \\ .15 & .15 & .10 & .01 & .22 \\ .06 & .07 & .36 & .15 & .35 \\ .27 & .08 & .07 & .41 & .09 \\ .23 & .19 & .36 & .24 & .10 \end{bmatrix}$$

$$I - C = \begin{bmatrix} .73 & -.39 & -.03 & -.02 & -.23 \\ -.15 & .85 & -.10 & -.01 & -.22 \\ -.06 & -.07 & .64 & -.15 & -.35 \\ -.27 & -.08 & -.07 & .59 & -.09 \\ -.23 & -.19 & -.36 & -.24 & .90 \end{bmatrix}$$

$$(I-C)^{-1} \cong \begin{bmatrix} 3.519 & 2.5190 & 2.1731 & 1.7441 & 2.5320 \\ 1.7987 & 2.7014 & 1.7372 & 1.3331 & 1.9289 \\ 2.8646 & 2.6351 & 4.2304 & 2.5500 & 3.2763 \\ 2.6675 & 2.2700 & 2.2211 & 3.3804 & 2.4384 \\ 3.1336 & 2.8734 & 3.2065 & 2.6486 & 4.1262 \end{bmatrix}$$

$$(I-C)^{-1}\vec{d} \cong \begin{bmatrix} 33.485 \\ 27.360 \\ 44.285 \\ 38.523 \\ 45.542 \end{bmatrix} \begin{matrix} \text{agriculture} \\ \text{manufacturing} \\ \text{trade} \\ \text{services} \\ \text{energy} \end{matrix}$$

(b) The three biggest entries in the services column are TRADE, SERVICES, and ENERGY.

② (a) Let $\vec{x}$ and $\vec{y}$ be in the set and let α and β be scalars. Then

$$\alpha\vec{x} + \beta\vec{y} = \alpha\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \beta\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \alpha x_1 + \beta y_1 \\ \alpha x_2 + \beta y_2 \end{bmatrix}$$

It follows that

$$(\alpha x_1 + \beta y_1) + (\alpha x_2 + \beta y_2) = \alpha\underbrace{(x_1 + x_2)}_{\substack{=0 \\ \text{since} \\ \vec{x} \text{ is in} \\ \text{the set}}} + \beta\underbrace{(y_1 + y_2)}_{\substack{=0 \\ \text{since} \\ \vec{y} \text{ is in} \\ \text{the set}}}$$

$$= \alpha 0 + \beta 0$$

$$= 0.$$

Since $\alpha\vec{x} + \beta\vec{y}$ satisfies the set condition, $\alpha\vec{x} + \beta\vec{y}$ is in the set. Thus any linear combination of any two vectors in the set is also in the set.

$\Rightarrow$ SUBSPACE

(b)

$$\underset{\substack{\uparrow \\ \text{in set}}}{\begin{bmatrix} 3 \\ 0 \end{bmatrix}} + \underset{\substack{\uparrow \\ \text{in set}}}{\begin{bmatrix} 0 \\ 4 \end{bmatrix}} = \underset{\substack{\uparrow \\ \text{not} \\ \text{in set}}}{\begin{bmatrix} 3 \\ 4 \end{bmatrix}}$$

NOT A SUBSPACE

(c)

$$\underset{\substack{\uparrow \\ \text{in set}}}{\begin{bmatrix} 2 \\ -2 \end{bmatrix}} + \underset{\substack{\uparrow \\ \text{in set}}}{\begin{bmatrix} 3 \\ 3 \end{bmatrix}} = \underset{\substack{\uparrow \\ \text{not} \\ \text{in set}}}{\begin{bmatrix} 5 \\ 1 \end{bmatrix}}$$

NOT A SUBSPACE

(d) Same as (c)!

③ $\begin{bmatrix} 1 & -3 & 2 & -4 \\ -3 & 9 & -1 & 5 \\ 2 & -6 & 4 & -3 \\ -4 & 12 & 2 & 7 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & -3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

↳ This tells us that $\vec{v}_2$ is a multiple of $\vec{v}_1$. Removing $\vec{v}_2$ will leave us with three linearly ind. vectors that span the same space.

Answer:

$\begin{bmatrix} 1 \\ -3 \\ 2 \\ -4 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} -4 \\ 5 \\ -3 \\ 7 \end{bmatrix}$

dimension = 3

④ (a) $A\vec{x} = \vec{0} \Rightarrow \left[\begin{array}{cccc|c} 1 & 1 & -1 & 2 & 0 \\ 2 & 2 & 3 & 1 & 0 \\ -1 & -1 & 0 & -5 & 0 \end{array}\right]$

I omitted a negative sign!

$\xrightarrow{\text{RREF}} \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \Rightarrow \begin{array}{l} x_1 + x_2 = 0 \\ x_3 = 0 \\ x_4 = 0 \\ x_2 \text{ free} \end{array} \Rightarrow \vec{x} = \begin{bmatrix} -x_2 \\ x_2 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow \vec{x} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \text{null}(A) = \text{span}\left(\left\{\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}\right\}\right)$

(b) RREF tells us which cols of A form a basis for its col space

$\Rightarrow \text{col}(A) = \text{span}\left(\left\{\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -5 \end{bmatrix}\right\}\right)$

(c) rank = 3

④ (a) $A\vec{x}=\vec{0}$: $\left[\begin{array}{cccc|c} 1 & 1 & -1 & 2 & 0 \\ 2 & 2 & -3 & 1 & 0 \\ -1 & -1 & 0 & -5 & 0 \end{array}\right] \xrightarrow[\text{RREF}]{} \left[\begin{array}{cccc|c} 1 & 1 & 0 & 5 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

$$\Rightarrow \begin{cases} x_1 + x_2 \quad + 5x_4 = 0 \\ \qquad\quad x_3 + 3x_4 = 0 \end{cases} \Rightarrow \begin{array}{l} x_1 = -x_2 - 5x_4 \\ x_3 = -3x_4 \\ x_2, x_4 \text{ free} \end{array}$$

$$\Rightarrow \vec{x} = \begin{bmatrix} -x_2 - 5x_4 \\ x_2 \\ -3x_4 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -5 \\ 0 \\ -3 \\ 1 \end{bmatrix}$$

$$\Rightarrow \text{null}(A) = \text{span}\left(\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ -3 \\ 1 \end{bmatrix} \right\}\right).$$

(b) RREF tells us which columns of A form a basis for its column space $\longleftrightarrow$ the pivot columns:

$$\text{col}(A) = \text{span}\left(\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix} \right\}\right).$$

(c) $\text{rank}(A) = \dim(\text{col}(A)) = 2$.

(5) (a) By the Rank-Nullity Theorem,

$$\text{rank}(A) + \dim(\text{null}(A)) = 4.$$

Since the column space of A is not all of $\mathbb{R}^4$, the rank of A (which equals $\dim(\text{col}(A))$) is at most 3. It follows that the dimension of the null space of A must be at least 1.

(b) Since the dimension of the null space of A is 0, it follows that $A\vec{x} = \vec{b}$ has a unique solution for any $\vec{b}$, by the Invertible Matrix Theorem.

(c) Since B has linearly independent columns, the RREF of B will have a pivot in each column. This means $B\vec{x} = \vec{0}$ has no free variables, hence the null space of B has dimension 0.

(d) We need two pivot columns and two free variable columns. Ex:

$$\begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 4 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and anything row-equivalent.