Problem set 3

(1) Since $A\vec{x}=\vec{b}$ is consistent for every $\vec{b}$, it follows that the RRE form of A does not have a row of all zeros. (If it did, we could find a $\vec{b}$ such that $[A \,\vdots\, \vec{b}]$ row-reduced to a matrix with a row of the form $[0 \ 0 \ \cdots \ 0 \,\vdots\, \blacksquare]$, where $\blacksquare \neq 0$, which is inconsistent. For some $\vec{b}$'s, we might end up with a 0 in that last spot when row-reducing, which wouldn't be a problem, but not for all $\vec{b}$'s.) It follows that A has a pivot in every row. Since A is square, it then must have a pivot in every column, too. Thus $A\vec{x}=\vec{b}$ has no free variables and thus exactly one solution, for each $\vec{b}$ in $\mathbb{R}^n$. TRUE

(2) G = amount produced by Genosha
M = " " " Moreau
S = " " " Sodor

amount produced = amount consumed

$$G = \frac{1}{3}G + \frac{1}{2}M \qquad\qquad -\frac{2}{3}G + \frac{1}{2}M = 0$$
$$M = \frac{2}{3}G + \frac{1}{4}S \quad \Rightarrow \quad \frac{2}{3}G - M + \frac{1}{4}S = 0$$
$$S = \frac{1}{2}M + \frac{3}{4}S \qquad\qquad \frac{1}{2}M - \frac{1}{4}S = 0$$

$$\Rightarrow \left[\begin{array}{ccc|c} -2/3 & 1/2 & 0 & 0 \\ 2/3 & -1 & 1/4 & 0 \\ 0 & 1/2 & -1/4 & 0 \end{array}\right] \xRightarrow{\text{RREF}} \left[\begin{array}{ccc|c} 1 & 0 & -3/8 & 0 \\ 0 & 1 & -1/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow$ $G = 3/8\ S$
$M = 1/2\ S$
S free

$\Rightarrow$ Both Genosha + Moreau produce, and thus consume, a fraction of what Sodor consumes/produces.

SODOR

③ (a) in = out

$$800 = x_1 + x_3$$
$$x_1 + x_4 = x_2 + 200$$
$$x_2 + 100 = x_5 + 600$$
$$x_5 + 450 = x_7 + 400$$
$$x_7 + 600 = x_4 + x_6$$
$$x_3 + x_6 = 750$$

$$\Rightarrow \begin{aligned} x_1 \quad + x_3 \qquad\qquad &= 800 \\ x_1 - x_2 \quad + x_4 \qquad &= 200 \\ x_2 \quad - x_5 \quad &= 500 \\ x_5 \quad - x_7 &= -50 \\ x_4 \quad + x_6 - x_7 &= 600 \\ x_3 \quad + x_6 \quad &= 750 \end{aligned}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 800 \\ 1 & -1 & 0 & 1 & 0 & 0 & 0 & 200 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 500 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & -50 \\ 0 & 0 & 0 & 1 & 0 & 1 & -1 & 600 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 750 \end{bmatrix}$$

$$\underset{\text{RREF}}{\Rightarrow} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 & 0 & 50 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 & 450 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 750 \\ 0 & 0 & 0 & 1 & 0 & 1 & -1 & 600 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & -50 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{cases} x_1 = x_6 + 50 \\ x_2 = x_7 + 450 \\ x_3 = -x_6 + 750 \\ x_4 = -x_6 + x_7 + 600 \\ x_5 = x_7 - 50 \\ x_6, x_7 \text{ free} \end{cases}$$

(b) If $x_1 = 0$, then $x_6 = -50$, which can't happen.

NO

(4) $\vec{x}_k = \begin{bmatrix} \text{\% enrolled in year } k \\ \text{\% not enrolled in year } k \end{bmatrix}$

2nd col. shows where unenrolled employees go

$\vec{x}_{k+1} = P\vec{x}_k$ where $P = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix}$

1st column shows where the enrolled employees go

Solve $P\vec{q} = \vec{q}$

$(P-I)\vec{q} = 0$

$$\left[\begin{array}{cc|c} -.2 & .3 & 0 \\ .2 & -.3 & 0 \end{array}\right] \xrightarrow{R_1+R_2} \left[\begin{array}{cc|c} -.2 & .3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

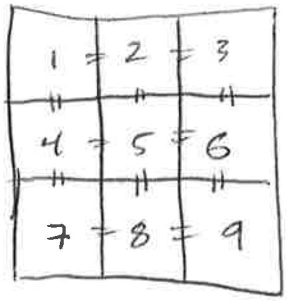
$$\xrightarrow{-10R_1} \left[\begin{array}{cc|c} 2 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{cases} 2q_1 = 3q_2 \\ q_2 \text{ free} \end{cases}$$

Since $1 = q_1 + q_2 = \frac{3}{2}q_2 + q_2 = \frac{5}{2}q_2$

$\Rightarrow q_2 = 2/5, \; q_1 = 3/5$

$\Rightarrow$ 60% enrolled, 40% unenrolled

(5)



$$P = \begin{bmatrix} 1/3 & 1/4 & 0 & 1/4 & 0 & 0 & 0 & 0 & 0 \\ 1/3 & 1/4 & 1/3 & 0 & 1/5 & 0 & 0 & 0 & 0 \\ 0 & 1/4 & 1/3 & 0 & 0 & 1/4 & 0 & 0 & 0 \\ 1/3 & 0 & 0 & 1/4 & 1/5 & 0 & 1/3 & 0 & 0 \\ 0 & 1/4 & 0 & 1/4 & 1/5 & 1/4 & 0 & 1/4 & 0 \\ 0 & 0 & 1/3 & 0 & 1/5 & 1/4 & 0 & 0 & 1/3 \\ 0 & 0 & 0 & 1/4 & 0 & 0 & 1/3 & 1/4 & 0 \\ 0 & 0 & 0 & 0 & 1/5 & 0 & 1/3 & 1/4 & 1/3 \\ 0 & 0 & 0 & 0 & 0 & 1/4 & 0 & 1/4 & 1/3 \end{bmatrix}$$

Solve $P\vec{q} = \vec{q}$

$P\vec{q} - I\vec{q} = \vec{0}$

$(P - I)\vec{q} = \vec{0}$

Augmented matrix :

$$\begin{bmatrix} -2/3 & 1/4 & 0 & 1/4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1/3 & -3/4 & 1/3 & 0 & 1/5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/4 & -2/3 & 0 & 0 & 1/4 & 0 & 0 & 0 & 0 \\ 1/3 & 0 & 0 & -3/4 & 1/5 & 0 & 1/3 & 0 & 0 & 0 \\ 0 & 1/4 & 0 & 1/4 & -4/5 & 1/4 & 0 & 1/4 & 0 & 0 \\ 0 & 0 & 1/3 & 0 & 1/5 & -3/4 & 0 & 0 & 1/3 & 0 \\ 0 & 0 & 0 & 1/4 & 0 & 0 & -2/3 & 1/4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/5 & 0 & 1/3 & -3/4 & 1/3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/4 & 0 & 1/4 & -2/3 & 0 \end{bmatrix}$$

$$\xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -4/3 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -4/3 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -5/3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -4/3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -4/3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$q_1 = q_9$

$q_2 = \frac{4}{3} q_9$

$q_3 = q_9$

$q_4 = \frac{4}{3} q_9$

$q_5 = \frac{5}{3} q_9$

$q_6 = 4/3\, q_9$

$q_7 = q_9$

$q_8 = 4/3\, q_9$

q_9 free

$$1 = q_1 + \dots + q_9 = \left(1 + \frac{4}{3} + 1 + \frac{4}{3} + \frac{5}{3} + \frac{4}{3} + 1 + \frac{4}{3} + 1\right) q_9$$

$$= 11 q_9 \Rightarrow q_9 = 1/11 \Rightarrow \boxed{q_1 = 1/11}$$

$\cancel{\Rightarrow \vec{q} = 1/11}$