Math 194
Problem Set 5

1. Suppose A and B are $n \times n$ matrices and that the matrices B and AB are invertible. Show that A is invertible. (Note: You cannot say that $(AB)^{-1} = B^{-1}A^{-1}$, since we don't know that A is invertible yet.)

2. Let
$$A = \begin{bmatrix} 2 & 1 & 1 \\ 6 & 4 & 5 \\ 4 & 1 & 3 \end{bmatrix}.$$
   (a) Find elementary matrices E, F, and G such that $GFEA = U$, where U is an upper triangular matrix (that is, all the entries of U below the main diagonal are 0).
   (b) Find $L = E^{-1}F^{-1}G^{-1}$. What kind of matrix is L?

3. Find the inverse of the matrix $A = \begin{bmatrix} -1 & -3 & -3 \\ 2 & 6 & 1 \\ 3 & 8 & 3 \end{bmatrix}$ by hand (without using Wolfram Alpha or any other similar tool).

4. Let A be a 3×3 matrix and suppose that
$$2\mathbf{a}_1 + \mathbf{a}_2 - 4\mathbf{a}_3 = \mathbf{0}.$$
   Is A invertible? Justify your answer.

5. Suppose that B is a 3×4 matrix with the property that $3\mathbf{b}_1 - \mathbf{b}_2 + 5\mathbf{b}_3 + 7\mathbf{b}_4 = \mathbf{0}$, and suppose that A is a 3×3 matrix. Show that the columns of AB are linearly dependent.

6. Determine whether each of the following sets of vectors is linearly independent. Justify your answers.
   (a) $\begin{bmatrix} 5 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 12 \\ 1 \\ -1 \\ -5 \end{bmatrix}$
   (b) $\begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ 5 \\ -2 \end{bmatrix}, \begin{bmatrix} -3 \\ 5 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ -4 \\ 7 \end{bmatrix}$
   (c) $\begin{bmatrix} 5 \\ 10 \\ 15 \\ 0 \end{bmatrix}, \begin{bmatrix} 10 \\ 20 \\ 30 \\ 5 \end{bmatrix}$
   (d) $\begin{bmatrix} a \\ 2a \\ a \end{bmatrix}, \begin{bmatrix} 3b \\ b \\ b \end{bmatrix}, \begin{bmatrix} 2a+3b \\ 4a+b \\ 2a+b \end{bmatrix}$, where a and b are particular constants
   (e) $\begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ -24 \\ 11 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ -1 \end{bmatrix}$

7. Suppose that $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ form a linearly independent set of vectors in $\mathcal{R}^3$. Suppose also that the vector $\mathbf{u} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3$ and that $\mathbf{u} = b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + b_3\mathbf{v}_3$. True or False: It must be the case that $a_1 = b_1$, $a_2 = b_2$, and $a_3 = b_3$. Justify your answer.

8. Let A be a 4×4 matrix with reduced row echelon form given by $U = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. If the first column of A is $\begin{bmatrix} -3 \\ 5 \\ 2 \\ 1 \end{bmatrix}$ and the second column of A is $\begin{bmatrix} 4 \\ -3 \\ 7 \\ -1 \end{bmatrix}$, determine the third and fourth columns.