Math 194
Problem Set 2

1. Solve the following system of linear equations *by hand* using Gaussian elimination. You must show every important step in order to receive full credit. (Hint: If the numbers get really messy, you're doing it wrong.)

$$\begin{array}{rcrcrcrcr} -2x_1 & + & 2x_2 & + & 6x_3 & + & 8x_4 & = & -4 \\ -2x_1 & - & 3x_2 & + & 16x_3 & - & 7x_4 & = & -9 \\ 3x_1 & - & x_2 & - & 13x_3 & - & 6x_4 & = & 8 \\ 4x_1 & + & x_2 & - & 22x_3 & - & x_4 & = & 13 \end{array}$$

Please note that you must do the row reduction for the above problem by hand. For the following problems, you're welcome to use WolframAlpha to assist you. Please make a note on your work where you use WolframAlpha.

2. Using the notation introduced in class (where $\blacksquare$ denotes a pivot position and $*$ can be any real number), each matrix below represents the row echelon form of the *augmented matrix* for a system of linear equations. In each case, determine if the system has infinitely many solutions, exactly one solution, or no solutions.

   (a) $\begin{bmatrix} \blacksquare & * & * \\ 0 & 0 & \blacksquare \\ 0 & 0 & 0 \end{bmatrix}$

   (b) $\begin{bmatrix} \blacksquare & * & * & * \\ 0 & 0 & \blacksquare & * \end{bmatrix}$

   (c) $\begin{bmatrix} \blacksquare & * & * \\ 0 & \blacksquare & 0 \\ 0 & 0 & 0 \end{bmatrix}$

   (d) $\begin{bmatrix} 0 & \blacksquare & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * \end{bmatrix}$

3. Suppose A is a 3×4 matrix and $\mathbf{b}$ is a vector in $\mathbb{R}^3$.

   (a) Could the matrix equation $A\mathbf{x} = \mathbf{b}$ have *no solutions*? If so, construct a matrix A and vector $\mathbf{b}$ that has this property. If not, explain why not.

   (b) Could the matrix equation $A\mathbf{x} = \mathbf{b}$ have *exactly one solution*? If so, construct a matrix A and vector $\mathbf{b}$ that has this property. If not, explain why not.

   (c) Could the matrix equation $A\mathbf{x} = \mathbf{b}$ have *infinitely many solutions*? If so, construct a matrix A and vector $\mathbf{b}$ that has this property. If not, explain why not.

4. Given the vectors defined below, determine if $\mathbf{b}$ is a linear combination of the vectors $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$. If it is, provide the coefficients for the linear combination. If it is not, explain why not.

$$\mathbf{a}_1 = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}, \mathbf{a}_2 = \begin{bmatrix} -2 \\ 5 \\ 4 \end{bmatrix}, \mathbf{a}_3 = \begin{bmatrix} 2 \\ 14 \\ -6 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 19 \\ 0 \\ -13 \end{bmatrix}$$

5. Construct a 3×3 matrix A with *all nonzero entries* and a vector $\mathbf{b}$ in $\mathbb{R}^3$ such that $\mathbf{b}$ *cannot* be written as a linear combination of the columns of A.

6. Give a geometric description (point, line, plane, all of $\mathbb{R}^3$) of Span$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ for the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 6 \\ -5 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -2 \\ 4 \\ -3 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 4 \\ 6 \\ -10 \end{bmatrix}.$$

   Justify your answer.

7. Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix}$. Which of the following sets has the same span as the set of all three vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$? (Justify your answers.)

   (a) $\{\mathbf{v}_1, \mathbf{v}_2\}$

   (b) $\{\mathbf{v}_2, \mathbf{v}_3\}$

   (c) $\{\mathbf{v}_1, \mathbf{v}_3\}$

8. Suppose A is an $n \times n$ matrix with the property that $A\mathbf{x} = \mathbf{0}$ has only the trivial solution $\mathbf{x} = \mathbf{0}$. Is it true that the matrix equation $A\mathbf{x} = \mathbf{b}$ is consistent for any vector $\mathbf{b}$ in $\mathbb{R}^n$?